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Steady Cylindrical Expansion of a Monatomic Gas into Vacuum

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The purpose of this Note is to re-examine certain aspects of steady flows into vacuum with cylindrical symmetry, the original treatment being given in Ref. 1. Previous theoretical work on the subject has dealt with the problem via certain models of the Boltzmann equation of the B-G-K type. Edwards and Cheng[1] and, in a more sophisticated way, Hamel and Willis[2] both followed this line of attack and in this respect the present Note says nothing new. The approximation procedures adopted in these papers were put on a firmer base by Freeman[3] by use of the method of matched asymptotic expansions. However, for the case of Maxwell molecules with cylindrical symmetry, the straightforward application of this method to the Boltzmann equation appeared to fail. It is the purpose herein to examine this case and to deduce an approximation procedure that will produce a uniformly valid solution for large r. In doing so it is of theoretical interest to see how the Chapman-Enskog expansion provides us with this uniform approximation.

The Boltzmann equation for steady flow with cylindrical symmetry can be written in nondimensional form as

$$\xi \frac{\partial f}{\partial r} + \frac{\eta^2}{r}\frac{\partial f}{\partial \xi} - \frac{\xi\eta}{r}\frac{\partial f}{\partial \eta} = AnT^{\beta}(F - f) \qquad (1)$$

where (r,θ,z) are cylindrical coordinates and (ξ,η,ζ) are components of molecular velocity in the directions of increasing (r,θ,z). f is the distribution function and F the local Maxwellian. For the moment, we will be dealing with the B-G-K collision model with a collision frequency dependent on temperature.[4] Also we are dealing with flows from near continuum sources and so in suitably nondimensionalized variables A is proportional to the inverse source Knudsen number and is assumed large. For $\beta \neq 0$ (non-Maxwell molecules) the problem can be approached using the method of matched asymptotic expansions; an inner (continuum) expansion breaks down when $r = 0(A^{(3/2)\beta})$, and an outer solution can be constructed[2] enabling the whole solution to be described. The region of validity of the inner expansion is found by balancing both sides of Eq. (1). This is equivalent to examining the Chapman-Enskog expansion for the Boltzmann equation and in particular its uniformity for large r. The two approaches are identical.

For $\beta = 0$, the case of Maxwell molecules, the full Boltzmann equation is modelled exactly, at least to the extent relevant to this Note, by Eq. (1). However, in this case the aforementioned procedure cannot be followed because the collision terms in (1) will never become as large as the convective terms. Nevertheless it can be shown via the Navier-Stokes equations that there is a logarithmic singularity at infinity in an expansion about the inviscid solution; therefore, how do we construct a uniformly valid solution for large r? The answer centers on the distinction between the continuum and inviscid solutions, and it will be shown that although the continuum solution (the Chapman-Enskog expansion) is uniformly valid, the inviscid solution (the Euler expansion) is not.

The Chapman-Enskog solution of the Boltzmann equation involves an asymptotic expansion of the distribution function in powers of some reference Knudsen number. The variables in this solution are the full temperature, density, and mass velocity of the gas, all of which depend on the reference Knudsen number itself. It is this choice of these variables which makes the Chapman-Enskog solution uniformly valid in this problem. In this context the true zeroth-order approximation would be the inviscid solution that is obtained by expanding all the thermodynamic variables in powers of the reference Knudsen number, keeping the radial variable of order one; it is this solution that is not uniformly valid. It is apparent in this problem that a correct zeroth-order approximation is obtained for the distribution function by providing a Maxwellian distribution with the correct parameters in the form of temperature, density and gas velocity. In view of these observations the method of strained coordinates will be shown to be a natural method of approach in finding a uniformly valid approximation to the distribution function and thermodynamic variables.

The Chapman-Enskog solution for this problem can be written

$$f = f_0(T,n,u)[1 + 0(1/A)]$$

where

$$T = T(r,A) \qquad (2)$$
$$n = n(r,A)$$

and

$$u = u(r,A)$$

are the temperature, density, and gas velocity. The variables have been nondimensionalized with respect to inviscid sonic values and A^{-1} is the Knudsen number arising from this scaling. r is the radial coordinate and f_0 is the Maxwellian distribution with T, n, and u as parameters. For the sake of brevity the details of the first-order term in Eq. (2) are omitted, but it is these terms that give rise to the Navier-Stokes equations when moments of the full distribution function are taken. These transport equations can be written

$$u^2\frac{du}{dx} + \frac{2}{5}\left\{u\frac{dT}{dx} - uT - T\frac{du}{dx}\right\} = \frac{4}{3}A^{-1}u^2\left\{T\frac{d^2u}{dx^2} + \frac{du}{dx}\cdot\frac{dT}{dx} - uT - \frac{u}{2}\cdot\frac{dT}{dx}\right\} + 0(A^{-2})$$

$$u\frac{dT}{dx} + \frac{2Tu}{3} + \frac{2T}{3}\frac{du}{dx} = \frac{5}{3}A^{-1}u\left\{\frac{4}{3}T\left[\left(\frac{du}{dx}\right)^2 - u\frac{du}{dx} + u^2\right] + T\frac{d^2T}{dx^2} + \left(\frac{dT}{dx}\right)^2\right\} + 0(A^{-2}) \qquad (3)$$

where $x = \log r$. The density and pressure have been eliminated using the continuity and perfect gas relations. In these equations the temperature and gas velocity are functions of r and A as in Eq. (2).

The inviscid solution is the zeroth-order term obtained by expanding, n, T, and u in powers of A^{-1}. If the solution so constructed is denoted by a subscript 0, then the temperature, density, and velocity take the familiar implicit form

$$T_0 = 2(1 - u_0^2/4)$$
$$n_0 = (4/3)^{3/2}(1 - u_0^2/4)^{3/2} \qquad (4)$$

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and

$$e^{-x} = (4/3)^{3/2}u_0(1 - u_0^2/4)^{3/2}$$

An expansion scheme of this type has interesting consequences when introduced into Eqs. (2). For the region of interest ($r \to \infty$) it transpires that the $0(A^{-1})$ term in the Chapman-Enskog expansion does not produce a nonuniformity. However, making an estimate of the error terms in the inviscid expansion using the Navier-Stokes equations, it may be shown for $r \to \infty$ that

$$f_0(T,n,u) = f_0(T_0,n_0,u_0)\{1 + A^{-1}[G \log r + 0(1)] + 0(A^{-2})\} \quad (5)$$

where G is $0(1)$ as $r \to \infty$. Here the function $f_0(T_0,n_0,u_0)$ is just the Maxwellian distribution corresponding to the inviscid solution. It is now apparent that although the Chapman-Enskog solution $f_0(T,n,u)$ is uniformly valid, the inviscid solution $f_0(T_0,n_0,u_0)$ is not due to the logarithmic singularity at infinity occurring in Eq. (5). The purpose of the following work is to determine suitable functions $\tau_0(r,A)$, $\rho_0(r,A)$, and $V_0(r,A)$ so that $f_0(\tau_0,\rho_0,V_0)$ is a uniformly valid approximation to f as $r \to \infty$.

We return to the Navier-Stokes equations and slightly strain the coordinate x. We write

$$x = s + A^{-1}X_1(s) + A^{-2}X_2(s) + \ldots \quad (6)$$

together with the expansions

$$\begin{aligned} T(x,A) &= \tau_0(s) + A^{-1}\tau_1(s) + \ldots \\ u(x,A) &= V_0(s) + A^{-1}V_1(s) + \ldots \end{aligned} \quad (7)$$

Inserting these into Eqs. (3) we have the zeroth order in A^{-1}

$$\begin{aligned} V_0^2 \frac{dV_0}{ds} + \frac{2}{5}\left\{V_0 \frac{d\tau_0}{ds} - \tau_0 \frac{dV_0}{ds} - V_0\tau_0\right\} &= 0 \\ V_0 \frac{d\tau_0}{ds} + \frac{2}{3}\left\{\tau_0 V_0 + \tau_0 \frac{dV_0}{ds}\right\} &= 0 \end{aligned} \quad (8)$$

which are of course the inviscid equations with the strained coordinate s as the independent variable. To first order

$$\left(V_0^2 - \frac{2\tau_0}{5}\right)\frac{dV_1}{ds} + \frac{2}{5}V_0\frac{d\tau_1}{ds} + \left(2V_0\frac{dV_0}{ds} + \frac{2}{5}\frac{d\tau_0}{ds} - \frac{2}{5}\tau_0\right)V_1 - \frac{2}{5}\left(V_0 + \frac{dV_0}{ds}\right)\tau_1 = \frac{4}{3}V_0^2\left\{\tau_0\frac{d^2V_0}{ds^2} + \frac{dV_0}{ds}\cdot\frac{d\tau_0}{ds} - V_0\tau_0 - \frac{V_0}{2}\frac{d\tau_0}{ds}\right\} + \frac{2}{5}\tau_0 V_0\frac{dX_1}{ds} \quad (9a)$$

$$\frac{2}{3}\tau_0\frac{dV_1}{ds} + V_0\frac{d\tau_1}{ds} + \left(\frac{d\tau_0}{ds} + \frac{2}{3}\tau_0\right)V_1 + \frac{2}{3}\left(V_0 + \frac{dV_0}{ds}\right)\tau_1 = \frac{5}{3}V_0\left\{\frac{4}{3}\tau_0\left[\left(\frac{dV_0}{ds}\right)^2 - V_0\frac{dV_0}{ds} + V_0^2\right] + \tau_0\frac{d^2\tau_0}{ds^2} + \left(\frac{d\tau_0}{ds}\right)^2\right\} - \frac{2}{3}\tau_0 V_0\frac{dX_1}{ds} \quad (9b)$$

The straining of the variable x is now chosen to remove the terms in Eqs. (9) which produce the nonuniform behavior in the inviscid expansion. In Eq. (9a) it is the term $-\frac{4}{3}V_0^3\tau_0$ and in (9b) the term $\frac{20}{9}V_0^3\tau_0$. The choice

$$dX_1/ds = \tfrac{10}{3}V_0^2 \quad (10)$$

removes both these terms. Any possible nonuniformities revealed in the higher-order analysis can be removed by appropriate choices of X_2, X_3, etc. The solution of Eqs. (8) now gives us a uniformly valid solution as $r \to \infty$, this is

$$\begin{aligned} \tau_0 &= 2(1 - V_0^2/4) \\ \rho_0 &= (4/3)^{3/2}(1 - V_0^2/4)^{3/2} \end{aligned} \quad (11)$$

and

$$e^{-s} = (4/3)^{3/2}V_0(1 - V_0^2/4)^{3/2}$$

which are the inviscid solutions where the strained variable s has replaced x. The distribution function is given by

$$f_0 = [\rho_0/(2\pi\tau_0)^{3/2}] \exp -\{C^2/2\tau_0\} \quad (12)$$

where, on integration of Eq. (10)

$$x = s + \frac{20}{3A}\left\{(1 - V_0^2) - 3\log\left(\frac{4 - V_0^2}{3}\right)\right\} + 0\left(\frac{1}{A^2}\right) \quad (13)$$

X_1 vanishing when $V_0 = 1$. For $x \to \infty$, this can be inverted to give

$$s = x(1 - 40/3A) + 0(1/A^2) \quad (14)$$

neglecting exponentially small terms in s. Finally it will be noticed that the Maxwellian distribution furnishes a uniform approximation to the full solution as $r \to \infty$ and no anisotropy in the distribution function is apparent.

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A Numerical Method for the Solution of a Shell Problem

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THE set of two partial differential equations governing the static problem of a thin, elastic, and shallow shell[1,2] is

$$K\Delta\Delta w - \Delta_k F = q, \; Eh\Delta_k w + \Delta\Delta F = 0 \quad (1)$$

where $w(x,y)$ is the normal displacement of a point on the shell's middle surface, $F(x,y)$ stress function, E elastic modulus, h thickness of the shell, K flexural rigidity, and q the distributed load in the z direction. The symbol $\Delta\Delta$ stands for the biharmonic operator while Δ_k is defined as

$$\Delta_k(\;) = z_{,yy}(\;)_{,xx} - 2z_{,xy}(\;)_{,xy} + z_{,xx}(\;)_{,yy} \quad (2)$$

with (,) standing for differentiation. $z(x,y)$ defines the geometry of the shell's middle surface.

In absence of concentrated edge loads the boundary conditions for the boundary simply supported along the line with

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